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E. González–Arangüena
C. Manuel
D. Gómez
R. van den Brink

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Universidad Complutense de Madrid
Av. Puerta de Hierro, S/N
28040 Madrid
Tlf. 913944035
buc_est@buc.ucm.es

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E. González–Arangüena, C. Manuel, D. Gómez

Dpto. de Estadística e I.O. III. Escuela Universitaria de Estadística. Universidad Complutense de Madrid. Av. Puerta de Hierro s/n. 28040 Madrid (Spain).
email: {egaran, conrado, dagomez}@estad.ucm.es

R. van den Brink

Department of Econometrics and Tinbergen Institute. Free University. De Boelelaan 1105. 1081 HV Amsterdam (The Netherlands).
email: jrbrink@feweb.vu.nl

Abstract.

In this paper we introduce an extension of the model of restricted communication in cooperative games as introduced in Myerson (1977) by allowing communication links to be directed and the worth of a coalition to depend on the order in which the players enter the coalition. Therefore, we model the communication network by a directed graph and the cooperative game by a generalized characteristic function as introduced in Nowak and Radzik (1994). We generalize the Myerson value for undirected (or standard) communication situations to the context of directed communication and provide two axiomatizations of this digraph Myerson value using component efficiency and either fairness or the balanced contributions property.

Keywords. Myerson value, Digraph communication situations.

1 Introduction

A situation in which a finite set of players $N = \{1, \dots, n\}$ can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility* (or simply a TU-game). Besides the player set, a TU-game consists of a *characteristic function* assigning to every subset of players (coalition) a real number which represents the *worth* that these players can obtain by agreeing to cooperate.

In a TU-game there are no restrictions on the cooperation possibilities of the players, i.e., every coalition is feasible and can generate a worth. Restrictions on cooperation can arise from limiting communication possibilities between the players. This is a well-known topic introduced by Myerson (1977) that created a model in which players in a TU-game also are the nodes in an undirected communication graph. The idea, then, is that players can only cooperate and form a feasible coalition if its members are connected in the communication graph. As a solution, Myerson (1977) proposes to apply the *Shapley value* (Shapley, 1953) to a modified game in which the worth of a coalition equals the sum of the worths of its connected components. This solution is nowadays known as the *Myerson value*. Also, he gave an axiomatization of this solution in terms of component efficiency and fairness properties. Later, Myerson (1980) provides another axiomatization replacing fairness by balanced contributions.

The above mentioned model assumes that the communication links are symmetric, meaning that both players incident with the link have equal control over or access to the link. However, in many cases, communication is directed and thus the model based on undirected graphs is not suitable. This can be so, for example, because one of the players is able to initiate the communication or the relation but the other is not. In some occasions this is imposed by some regulatory authority. In other types of relations, as the network of citations of academic papers, it is obvious that each link has necessarily a fixed direction.

In order to model these situations, in this paper we identify a communication network with a directed graph. Moreover, we allow the worth of a coalition to depend on the order in which the players enter the coalition. Therefore, we apply *generalized characteristic functions* as introduced in Nowak and Radzik (1994). In a standard characteristic function any coalition or combination of players earns a certain worth. In a generalized characteristic function any *ordered* coalition or permutation of players earns a worth. It can be, for example, that the coalition that consists of players 1 and 2 earns a certain worth if first player 1 enters and then player 2, but earns a different worth if the players enter in the opposite order. So, a directed communication situation consists of a generalized TU-game and a directed graph on the player set. The model of Myerson can be seen as a directed communication situation where the worth of any permutation of a set of players is the same and, in the directed graph, between any pair of players there is either no arc or two arcs, one going in each direction. One of the main aims of the present paper is to generalize the model of undirected communication situations and the corresponding Myerson value to this new setting. To do this, we introduce a restricted game and a value that, for the case in which the directed communication situation is equivalent to an undirected one, yield the classical graph restricted game and the Myerson value.

The main issue that we must address is the way that the directed graph has an effect on the cooperation possibilities, i.e., what are the 'connected' or feasible coalitions in a directed graph. We call an ordered coalition connected if for every two consecutive players in the ordering there is a directed path from the predecessor to the successor (possibly containing players outside the coalition). Using this idea of connectedness, we assign to every directed communication situation a standard TU-game where the worth of any coalition depends on the dividends in the generalized characteristic function of all connected ordered subcoalitions. As solution for directed communication situations, we propose the Shapley value of this TU-game. This solution generalizes the Myerson value for undirected (or standard) communication situations as described in the previous paragraph.

We also generalize the axiomatizations using component efficiency and either fairness or the balanced contributions property.

Myerson (1977) also shows stability for superadditive games, meaning that creating a link between two players does not decrease their payoffs in case the game is superadditive. We show that this is not the case in our more general framework.

Another approach to the study of models in which restrictions in the communications are given by a directed graph can be found in Slikker and Van den Nouweland (2001). They characterized the so called allocation rules for directed communication situations which in their context consists of a set of players, a directed reward function, and a directed communication network. This directed reward function assigns a value to every possible directed communication network defined on the player set.

In Gilles, Owen and van den Brink (1992), van den Brink and Gilles (1996), Gilles and Owen (1994) and van den Brink (1997), the model of *games with a permission structure*, consisting of a (standard) TU-game and a directed graph is analyzed. They consider solutions applied to restricted games derived from the game and a directed graph, but that model does not generalize the Myerson restricted game nor the Myerson value for communication situations.

The remaining of this paper is organized as follows. After discussing some preliminaries on games, graphs and directed graphs in Section 2, in Section 3 we describe the model of directed communication and define connectedness yielding feasible coalitions and restricted games in these situations. In Section 4 we provide the two axiomatizations of the digraph Myerson value. Section

5 concludes with final remarks.

2 Preliminaries

2.1 Games and Generalized Games

Let $N = \{1, 2, \ldots, n\}$ be a finite set of players. A game in characteristic function form (a coalitional game or a TU-game) is a pair (N, v) where v (the characteristic function) is a real function defined on 2^N, the set of all subsets of N (coalitions), that satisfies $v(\emptyset) = 0$. For each $S \in 2^N$, $v(S)$ represents the (transferable) utility that players in S can obtain if they decide to cooperate. Implicitly, it is supposed that, if the players in S form a coalition, members of S must talk together and achieve a binding agreement.

We will denote by s the cardinality of the coalition $S \subset N$ and G^N will be the $2^n - 1$ dimensional vector space of TU-games with player set N. Its unanimity games basis $\{u_S\}_{\emptyset \neq S \subset N}$ is defined as follows:

$$\text{for all } S \subset N,\ S \neq \emptyset,\ u_S(T) = \begin{cases} 1, & S \subset T \\ 0, & \textit{otherwise.} \end{cases}$$

A *solution* for TU-games is a function which assigns a payoff vector $x \in \mathbb{R}^n$ to every TU-game in G^N. One of the most famous solutions is the *Shapley value* (Shapley (1953)), φ, which is given by:

$$\varphi_i(N, v) = \sum_{S \subset N \setminus \{i\}} \frac{(n-s-1)!s!}{n!} (v(S \cup \{i\}) - v(S)), \quad \text{for all } i \in N.$$

Nevertheless, in many social or economic situations, the formation of coalitions is a process in which not only the members of the coalitions are important but also the order in which they appear. Taking this idea into account, Nowak and Radzik (1994) introduced the concept of game in generalized characteristic function form.

For each $S \in 2^N \setminus \{\emptyset\}$, let us denote by $\pi(S)$ the set of all permutations or ordered coalitions of the players in S and, for notational convenience, $\pi(\emptyset) = \{\emptyset\}$. We will denote $\Omega(N) = \{T \in \pi(S) \mid S \subset N\}$ being the set of all ordered coalitions with players in N.

Given an ordered coalition $T \in \Omega(N)$, there exists $S \subset N$ such that $T \in \pi(S)$. We will denote $H(T) = S$ as the set of players in the ordered coalition T, and $t = |H(T)|$.

A game in generalized characteristic function form is a pair (N, v), N being the player set and v a real function (the generalized characteristic function), defined on $\Omega(N)$ and satisfying $v(\emptyset) = 0$.

For each $S \subset N$, and for every ordered coalition $T \in \pi(S)$, $v(T)$ represents the economic possibilities of the players in S if the coalition is formed following the order given by T. When there is no ambiguity with respect to the set of players N, we will identify the (generalized) game (N, v) with its (generalized) characteristic function v.

We will denote by $\mathcal{G}^N$ the set of all generalized cooperative games with players set N. $\mathcal{G}^N$ is a vector space with dimension $|\Omega(N)| - 1$. Let us observe that there exists an isomorphism between the vector space G^N and the subspace of $\mathcal{G}^N$ consisting of all games for which $v(T) = v(R)$ if $H(T) = H(R)$ holds. Intuitively, for games in G^N, the order in which the coalitions are formed is irrelevant.

Taking into account the previous idea we will sometimes identify each game $v \in G^N$ with the (transformed) game $\hat{v} \in \mathcal{G}^N$ defined by:

$$\hat{v}(T) = v(H(T)) \text{ for all } T \in \Omega(N).$$

Each ordered coalition $T = (i_1, \ldots, i_t) \in \Omega(N)$ establishes a strict linear order $\prec_T$ in $H(T)$, defined as follows:

for all $i, j \in H(T)$, $i \prec_T j$ (i precedes j in T) if and only if there exist $k, l \in \{1, \ldots, t\}$, $k < l$, such that $i = i_k$, $j = i_l$.

Given two ordered coalitions $T, R \in \Omega(N)$, we will denote $T \tilde{\subset} R$ if and only if:

a) $H(T) \subset H(R)$, and

b) $\forall\, i, j \in H(T)$, $i \prec_T j$ implies $i \prec_R j$.

Example 2.1 *Given* $N = \{1, 2, 3, 4\}$, $T = (3, 1)$, $R = (2, 3, 4, 1)$ *and* $P = (4, 1, 2, 3)$, $T \tilde{\subset} R$ *holds, but* $T \tilde{\subset} P$ *does not, as* $3 \prec_T 1$ *and* $1 \prec_P 3$.

In this paper, a special basis of $\mathcal{G}^N$, *the generalized unanimity basis*, consisting of the (generalized) unanimity games $\{w_T\}_{\emptyset \neq T \in \Omega(N)}$, will often be used. For any $T \in \Omega(N) \setminus \{\emptyset\}$, the generalized characteristic function w_T is defined as follows:

$$\text{for all } R \in \Omega(N),\ w_T(R) = \begin{cases} 1, \; if \;\; T \tilde{\subset} R \\ \\ 0, \; otherwise. \end{cases}$$

The transformed games $\{\hat{u_S}\}_{\emptyset \neq S \subset N}$ of the classical unanimity games $\{u_S\}_{\emptyset \neq S \subset N}$ of G^N, can be easily expressed in terms of the $\{w_T\}_{\emptyset \neq T \in \Omega(N)}$ in the following way:

$$\hat{u}_S = \sum_{T \in \pi(S)} w_T, \qquad \text{for each } S \in 2^N \setminus \{\emptyset\}.$$

For a given $v \in \mathcal{G}^N$, $\{\Delta_v^*(T)\}_{\emptyset \neq T \in \Omega(N)}$ is the set of the generalized unanimity coefficients of v (the coordinates of v in the generalized unanimity basis). Sánchez and Bergantiños (1997) proved that, for all $T \in \Omega(N) \setminus \{\emptyset\}$:

$$\Delta_v^*(T) = \sum_{R \widetilde{\subset} T} (-1)^{t-r} v(R).$$

In their seminal paper on games in generalized characteristic function form, Nowak and Radzik (1994), define a value for these games that generalizes the Shapley value for TU-games. Later, Sánchez and Bergantiños (1997) define and study another generalization of the Shapley value for TU-games to this class of generalized games, differing from the former in null player and symmetry axioms. A class of solutions including both previously mentioned is discussed in van den Brink et al. (2008)

Finally, given a game $v \in \mathcal{G}^N$, the *average game* $\overline{v} \in G^N$ is defined as follows:

$$\overline{v}(S) = \frac{1}{s!} \sum_{T \in \pi(S)} v(T), \qquad S \subset N.$$

2.2 Graphs and Directed Graphs

A graph is a pair (N, γ), $N = \{1, 2, \dots, n\}$ being a finite set of nodes and γ a collection of *links* (edges or ties), that is, unordered pairs $\{i, j\}$ with $i, j \in N$, $i \neq j$. A graph (N, δ) is a subgraph of (N, γ) if $\delta \subset \gamma$. We will denote by Γ^N the class of all graphs with nodes set N. When there is no ambiguity with respect to N, we will refer to the graph (N, γ) as γ.

If $\{i, j\} \in \gamma$, we will say that i and j are *directly connected* in γ. We will say that i and j are *connected* in γ if it is possible to join them by a sequence of edges from γ. We will say that a subset S of N is *connected* in (N, γ) if any pair of nodes i and j in S are connected in γ. (Note that this is different than the notion of connectedness in Myerson (1977), where a coalition S is connected if every pair of players in S are connected by a path only containing players that belong to that coalition.)

Given a graph (N, γ), the notion of connectivity induces a partition of N in connected components. Two nodes i and j, $i \neq j$, are in the same *connected component* if and only if they are connected. By connected component we mean what is also known as a maximal connected subset. Note that, although our definition of connectedness is different than that in Myerson (1977), we get the same components in a graph. N/γ denotes the set of all connected components in γ and more generally, for each $S \subset N$, S/γ is the set of all connected components in the *partial graph* $(S, \gamma_{|S})$, $\gamma_{|S}$ being the set of those links $\{i, j\} \in \gamma$ where both i and j are elements of S.

Given a graph (N, γ) and a subset $S \subset N$, we will note:

$$\mathcal{P}^{\gamma}(S) = \{U \subset S \mid U \neq \emptyset \textit{ and } U \textit{ connected in } (S, \gamma|_S)\}.$$

A directed graph or digraph is a pair (N, d), $N = \{1, 2, \ldots n\}$ being a set of nodes and d a subset of the collection of all ordered pairs (i, j), $i \neq j$, of elements of N. Each pair $(i, j) \in d$ is called an *arc*. In the following, if there is no ambiguity with respect to N, we will refer to the digraph (N, d) as d. We will denote D^N for the set of all possible digraphs with nodes set N.

3 The model

3.1 Connectedness in digraphs

In the following, the way that the directed graph has an effect on the connection possibilities, i.e., which are the connected sets in a directed graph, is of special importance.

Given a digraph (N, d), if $(i, j) \in d$, we will say that i is *directly connected* with j. Obviously, if i is directly connected with j, the reverse is not necessarily true. If i is not directly connected with j in the digraph, it may still be possible to connect them, provided that there are other nodes through which we can do so. We will say that i is *connected* with j in the digraph (N, d) if there is a directed path connecting them, i.e., if there exists an ordered sequence of nodes in N, $(i_1, i_2, \ldots, i_s)$, such that $i_1 = i$, $i_s = j$ and $(i_l, i_{l+1}) \in d$ for all $l \in \{1, 2, \ldots, s-1\}$.

To study directed communication we now come to the notion of connectedness of an ordered set in a directed graph. We will say that an ordered set $T = (i_1, i_2, \ldots, i_t) \in \Omega(N)$ is *connected* in the digraph (N, d) if, for all $l = 1, \ldots, t-1$, i_l is connected (not necessarily directly connected) with i_{l+1} in the digraph (N, d).[1]

For each subset $S \subset N$, we will denote:

$$\Omega^d(S) = \{T \in \Omega(S) \mid T \textit{ connected in } (S, d|_S)\},$$

where, for all $S \subset N$, $d|_S = \{(i, j) \in d \mid i, j \in S\}$.

Given a digraph $(N, d) \in D^N$, we can define the *induced graph* $(N, \gamma(d))$ in Γ^N as follows:

$$\gamma(d) = \{\{i, j\} \mid i, j \in N \textit{ and } [(i, j) \in d \;\; \textit{or} \;\; (j, i) \in d]\}$$

A (not ordered) set $C \subset N$ is a component in the digraph (N, d) if $C \in N/\gamma(d)$, i.e., if C is a connected component in the graph $(N, \gamma(d))$. So, given a digraph $(N, d) \in D^N$ we will establish a partition of N in components. We will denote by N/d the set of all the components of the directed graph (N, d). Obviously, $N/d = N/\gamma(d)$.

Let us observe that, given a component $C \in N/d$ and $T \in \Omega(C)$, it is possible

[1] This concept of connectedness is different from that of Amer, Giménez and Magaña (2007) who consider connected ordered sets in which each node is directly connected with its successor. Obviously, this connectedness concept does not generalize the classical one in the Myerson model.

that T is not an ordered connected set in (N,d).

Example 3.1 *Let* $N=\{1,2,3,4\}$, *and* $d=\{(1,2),(2,3),(3,2),(3,4)\}$ *as in Fig. 1. In this case:*

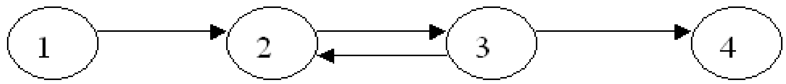

Fig. 1. Digraph in Example 3.1

i) Ordered coalitions $T=(1,2,4)$ *and* $R=(1,3,2,4)$ *are connected in* (N,d), *but* $T'=(1,4,3)$ *is not.*

ii) $\gamma(d)=\{\{1,2\},\{2,3\},\{3,4\}\}$, *and* $N/d=N/\gamma(d)=\{N\}$.

iii) Given $S=\{1,2,4\}\subset N$, $d|_S=\{(1,2)\}$*;* $S/d=\{\{1,2\},\{4\}\}$*;* $\Omega^d(S)=\Omega^{d|_S}(S)=\{(1),(2),(4),(1,2)\}$*;* $\gamma(d|_S)=\gamma(d)|_S=\{\{1,2\}\}$.

Given a graph $(N,\gamma)\in\Gamma^N$, let us denote by $(N,d(\gamma))$ the digraph containing, for each link $\{i,j\}\in\gamma$, two arcs (i,j) and (j,i). The map that assigns to each graph (N,γ) the digraph $(N,d(\gamma))$ is a bijection between Γ^N and its image set in D^N. In this sense, Γ^N is a subset of D^N.

Let us observe that, for any graph $(N,\gamma)\in\Gamma^N$ and for each $C\in N/\gamma$:

$$\Omega^{d(\gamma)}(C)=\Omega(C) \text{ holds,}$$

that is, all ordered subsets $T\in\pi(S)$ with $S\subset C$ are connected in the digraph $(N,d(\gamma))$.

A directed communication situation [2] (or digraph communication situation) is a triplet (N, v, d) where $(N, v) \in \mathcal{G}^N$ is a generalized TU-game and $(N, d) \in D^N$ is a directed graph. No particular relation is assumed between the game v and the digraph d other than the nodes in the digraph being the players in the game. This definition generalizes the well-known one of a communication situation (N, v, γ), in which $(N, v) \in G^N$ and $(N, \gamma) \in \Gamma^N$. We will denote by $\mathcal{CS}^N$ and $\mathcal{DCS}^N$ the sets of all (standard) communication situations and directed communication situations with nodes set N respectively.

We can associate to every digraph communication situation (N, v, d) a (standard) cooperative game $(N, v^d) \in G^N$ which incorporates both, the possible gains from cooperation (influenced by the coalitional order) as modeled by the generalized cooperative game (N, v) and the restrictions in the communications, reflected by the digraph (N, d).

In order that the definition for (N, v^d) can be easily understood, let us consider the idea of a game with restrictions in communication as it was introduced by Myerson (1977). For $(N, v, \gamma) \in \mathcal{CS}^N$, he defined the graph-restricted game $v^\gamma \in G^N$ as the game with characteristic function:

$$v^\gamma(S) = \sum_{C \in S/\gamma} v(C), \quad for\ all\ S \subset N.$$

The previous definition can be rewritten in terms of the dividends of game (N, v) as follows:

[2] This denomination is used in Slikker and Van den Nouweland (2001) with a different meaning. Nevertheless we have preferred to maintain it in order to emphasize the generalization of the classical concept of communication situation to this new setting of directed graphs.

$$v^{\gamma}(S) = \sum_{U \in \mathcal{P}^{\gamma}(S)} \Delta_v(U) = \sum_{R \in \Omega^{d(\gamma)}(S)} \frac{1}{r!} \Delta_v(H(R)). \tag{1}$$

The first equality in (1) holds because:

$$\forall\ C \in S/\gamma,\ v(C) = \sum_{\emptyset \neq\ U \subset C} \Delta_v(U) = \sum_{U \in \mathcal{P}^{\gamma}(C)} \Delta_v(U).$$

as $C \in S/\gamma$ and $\emptyset \neq U \subset C$ implies U connected in $(S, \gamma|_S)$, and obviously: U is connected in $(S, \gamma|_S)$ if and only if there exists $C \in S/\gamma$ such that $U \subset C$.

The last equality in (1) holds because $R \in \Omega^{d(\gamma)}(S)$ if and only if there exists $C \in S/\gamma$ such that $U = H(R) \subset C$. Then, for each one of the $r!$ ordered coalitions $R \in \pi(U)$, $\Delta_v(H(R)) = \Delta_v(U)$ holds.

Generalizing this idea, given $(N, v, d) \in \mathcal{DCS}^N$, we will define the characteristic function of $(N, v^d) \in G^N$. Suppose that in $(N, v, d) \in \mathcal{DCS}^N$, each coalition S can only obtain the dividends $\Delta_v^*(R)$ of its ordered subcoalitions $R \in \Omega(S)$ that are connected in the (partial) digraph $(S, d|_S)$. So, we define:

$$v^d(S) = \sum_{R \in \Omega^d(S)} \frac{1}{r!} \Delta_v^*(R), \qquad S \subset N.$$

We can consider $v^d(S)$ to be the expected dividend of all subcoalitions in S. For each subcoalition of S, this expectation is computed as the average of the $r!$ quantities:

$$\begin{cases} \Delta_v^*(R),\ if\ \ R \in \pi(H(R)) \cap \Omega^d(S) \\ \\ \quad 0, \quad\ if R \in \pi(H(R)) \setminus \Omega^d(S). \end{cases}$$

As, for each $S \subset N$, $\Omega^d(S) = \bigcup_{C \in S/d} \Omega^d(C)$ holds, we can rewrite the previous definition as follows:

$$v^d(S) = \sum_{C \in S/d} v^d(C) = \sum_{C \in S/d} \ \sum_{R \in \Omega^d(C)} \frac{1}{r!} \Delta_v^*(R), \qquad \text{for all } S \subset N.$$

In the next proposition we obtain a different expression for $v^d(S)$, $S \subset N$. This new expression (which can be considered as an alternative definition), establishes a direct relation between the characteristic functions of the games $(N, v) \in \mathcal{G}^N$ and $(N, v^d) \in G^N$.

Proposition 3.1 *Let $(N, v, d) \in \mathcal{DCS}^N$. Then, the characteristic function of game (N, v^d) is given by:*

$$v^d(S) = \sum_{C \in S/d} \sum_{R \in \Omega^d(C)} \lambda^d(R) v(R), \quad for\ each\ S \subset N, \tag{2}$$

where, for each $R \in \Omega^d(C)$, $\lambda^d(R) = \sum\limits_{T \in \Omega^d(C),\ T \tilde{\supset} R} \dfrac{(-1)^{t-r}}{t!}$.

Proof 3.1 *From the definition of the game v^d:*

$$v^d(S) = \sum_{C \in S/d} v^d(C) = \sum_{C \in S/d} \sum_{T \in \Omega^d(C)} \frac{\Delta_v^*(T)}{t!} =$$

$$= \sum_{C \in S/d} \sum_{T \in \Omega^d(C)} \sum_{T \tilde{\supset} R} \frac{(-1)^{t-r}}{t!} v(R) = \sum_{C \in S/d} \sum_{R \in \Omega^d(C)} \lambda^d(R) v(R). \quad \square$$

We will prove that, if we restrict ourselves to communication situations (in which case the game is equivalent to a standard TU-game and the digraph is equivalent to a graph), then the defined digraph restricted game coincides with the Myerson one. To do this, let us first obtain the expression of the restricted game when in the directed communication situation the digraph is a graph but the game is a generalized one.

Proposition 3.2 *Let (N, γ) be a graph in Γ^N and consider the digraph communication situation $(N, v, d(\gamma)) \in \mathcal{DCS}^N$. Then, for all $S \subset N$:*

$$v^{d(\gamma)}(S) = \sum_{C \in S/d(\gamma)} \sum_{R \in \pi(C)} \frac{v(R)}{r!} = \overline{v}^{\gamma}(S). \tag{3}$$

Proof 3.2 *From Proposition 3.1, for all $S \subset N$ we have:*

$$v^{d(\gamma)}(S) = \sum_{C \in S/d(\gamma)} \sum_{R \in \Omega^{d(\gamma)}(C)} \lambda^{d(\gamma)}(R) v(R).$$

Again, as (N, γ) is a graph, if $C \in S/d(\gamma)$, then $\Omega^{d(\gamma)}(C) = \Omega(C)$ holds.

It is obvious that, for $C \in S/d(\gamma)$, if $R \in \pi(C)$ then $\lambda^{d(\gamma)}(R) = \frac{1}{r!}$.

On the other hand, if $R \in \Omega(C) \setminus \pi(C)$ we have:

$$\lambda^{d(\gamma)}(R) = \sum_{T \in \Omega(C),\ T \widetilde{\supset} R} \frac{(-1)^{t-r}}{t!} = \sum_{m=0}^{c-r} \binom{c-r}{m} \binom{r+m}{m} m! \frac{(-1)^{r+m-r}}{(r+m)!} =$$

$$= \sum_{m=0}^{c-r} \binom{c-r}{m} \frac{1}{r!} (-1)^m = 0.$$

Then, for all $S \subset N$:

$$v^{d(\gamma)}(S) = \sum_{C \in S/d(\gamma)} \sum_{R \in \pi(C)} \frac{v(R)}{r!}.$$

The last equality in (3) trivially holds because of the definition of $\overline{v}$. $\square$

The next proposition gives the particular expression of the characteristic function of (N, v^d) for digraph communication situations in which the digraph is a graph and the game is a TU-game.

Proposition 3.3 *Let $(N, v, \gamma) \in \mathcal{CS}^N$ and consider the digraph communication situation $(N, \hat{v}, d(\gamma))$ with $\hat{v}(R) = v(H(R))$ for all $R \in \Omega(N)$. Then, $\hat{v}^{d(\gamma)} = v^\gamma$ holds.*

Proof 3.3 *From Proposition 3.2, $\hat{v}^{d(\gamma)} = \left(\overline{\hat{v}}\right)^\gamma$. On the other hand, it is obvious that, for all $v \in G^N$, $\overline{\hat{v}} = v$. Thus, the proposition holds.* $\square$

An allocation rule for digraph communication situations with nodes-players set N is a function $\Psi : \mathcal{DCS}^N \to I\!R^n$ that associates a vector of payoffs to each digraph communication situation.

Different allocation rules can be obtained by applying different solution concepts to the game (N, v^d). In this paper, we restrict ourselves to φ, the Shapley value (Shapley, 1953). For communication situations $(N, v, \gamma) \in \mathcal{CS}^N$, the Shapley value of game (N, v^γ) is known in the literature as the Myerson value. Because of previous propositions, given $(N, v, d) \in \mathcal{DCS}^N$, the Shapley value of (N, v^d) is a natural extension of the Myerson value for digraph communication situations. Then, we have the following definition.

Definition 3.1 *The Myerson value for directed communication situations (from now on, digraph Myerson value) is the allocation rule μ defined as follows:*

$$\mu(N, v, d) = \varphi(N, v^d), \quad for\ every\ (N, v, d) \in \mathcal{DCS}^N.$$

The first interesting property of this allocation rule is its additivity.

Definition 3.2 *An allocation rule $\Psi : \mathcal{DCS}^N \longrightarrow I\!R^n$ is additive if, for all (N, v, d), $(N, w, d) \in \mathcal{DCS}^N$, $\Psi(N, v + w, d) = \Psi(N, v, d) + \Psi(N, w, d)$.*

Proposition 3.4 *The digraph Myerson value is additive.*

Proof 3.4 *This is straightforward taking into account the additivity of the Shapley value and the fact that, if (N, v, d), $(N, w, d) \in \mathcal{DCS}^N$, then $(v+w)^d = $ $= v^d + w^d$.* $\square$

Example 3.2 *Let $N = \{1, 2, 3, 4\}$, $d = \{(1, 2), (1, 3), (2, 4), (3, 4)\}$, as in Fig. 2 and $v \in \mathcal{G}^N$ be defined as follows: $v = \sum_{i,j \in N,\ i \neq j} w_{(i,j)}$.*

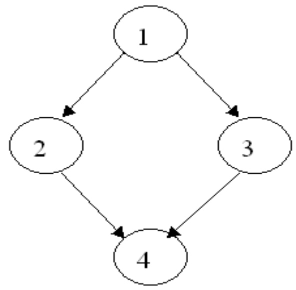

Fig. 2. Digraph in Example 3.2

Taking into account that $v^d = \sum_{i,j\in N,\ i\neq j} w^d_{(i,j)}$ *and*

$$w^d_{(i,j)} = \begin{cases} \frac{1}{2!}u_{\{i,j\}}, & if\ (i,j)\in d \\ \frac{1}{2!}\left[u_{\{1,2,4\}} + u_{\{1,3,4\}} - u_{\{1,2,3,4\}}\right], & if\ (i,j) = (1,4) \\ 0, & otherwise \end{cases}$$

we have:

$$\mu(N,v,d) = \varphi(N,v^d) = \sum_{i,j\in N,\ i\neq j} \varphi(N,w^d_{(i,j)}) = \frac{1}{2!}[\varphi(u_{\{1,2\}}) + \varphi(u_{\{1,3\}}) +$$
$$+\varphi(u_{\{1,3\}}) + \varphi(u_{\{2,4\}}) + \varphi(u_{\{3,4\}}) + \varphi(u_{\{1,2,4\}}) + \varphi(u_{\{1,3,4\}}) - \varphi(u_{\{1,2,3,4\}})] =$$
$$= \frac{1}{2!}\left[\left(\frac{1}{2},\frac{1}{2},0,0\right) + \left(\frac{1}{2},0,\frac{1}{2},0\right) + \ldots + \left(\frac{1}{3},0,\frac{1}{3},\frac{1}{3}\right) - \left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right] =$$
$$= \left(\frac{17}{24},\frac{13}{24},\frac{13}{24},\frac{17}{24}\right).$$

4 Two characterizations of the digraph Myerson value

This section is devoted to characterize the digraph Myerson value from its properties: *component efficiency, fairness* and *balanced contributions*.

An allocation rule for digraph communication situations satisfies the component efficiency property if the total payoff for the members of a component equals its worth under the restrictions in the communication. Formally,

Definition 4.1 *An allocation rule* $\Psi : \mathcal{DCS}^N \to I\!R^n$ *is component efficient if, for all* $(N, v, d) \in \mathcal{DCS}^N$ *and all* $C \in N/d$*, it holds that:*

$$\sum_{i \in C} \Psi_i(N, v, d) = v^d(C).$$

Remark 4.1 *The previous definition extends the classical one for standard communication situations. This is obvious from the fact that, for* $(N, v, \gamma) \in \mathcal{CS}^N$*,* $v^\gamma(C) = v(C)$ *if* $C \in N/\gamma$*.*

The fairness property for digraph communication situations states that if the possibility for direct communication between two players i and j disappears, other things being equal (i.e., the network changes to another one in which the arc $l = (i, j)$ or the arc (j, i) is removed), then the payoffs of both players change by the same amount.

Definition 4.2 *An allocation rule* $\Psi : \mathcal{DCS}^N \to I\!R^n$ *is fair if, for all* $(N, v, d) \in \mathcal{DCS}^N$ *and all* $(i, j) \in d$*, the following holds:*

$$\Psi_i(N, v, d) - \Psi_i(N, v, d \setminus \{(i, j)\} = \Psi_j(N, v, d) - \Psi_j(N, v, d \setminus \{(i, j)\}).$$

Remark 4.2 *The classical definition of fairness for (graph) communication situations can be obtained applying the previous one in a two step process. This is so because a link* $\{i, j\}$ *can be viewed as the pair of arcs* (i, j) *and* (j, i)*. However, note that fairness as defined above is stronger than applying Myerson's fairness with respect to deleting the two arcs at the same time, since that allows differences in the change of payoffs when deleting each arc but in the two step process these differences should be of opposite sign and equal in absolute value.*

It turns out that the above two properties characterize the digraph Myerson value.

Theorem 4.1 *The digraph Myerson value is the unique allocation rule* $\Psi : \mathcal{DCS}^N \to I\!R^n$ *satisfying component efficiency and fairness.*

Proof 4.1 *In order to prove component efficiency, let* $(N, v, d) \in \mathcal{DCS}^N$ *and* $C \in N/d$*. Then:*

$$\sum_{i\in C} \mu_i(N,v,d) = \sum_{i\in C} \mu_i(C, v|_C, d|_C) = \sum_{i\in C} \varphi_i\left(C, (v|_C)^{d|_C}\right) =$$

$$= \sum_{i\in C} \varphi_i\left(C, (v|_C)^{d}\right) = v^d(C).$$

The last equality holds because of the efficiency of the Shapley value.

Turning now to the fairness property of the digraph Myerson value, suppose that $(N, v, d) \in \mathcal{DCS}^N$ *and let* $i, j \in N$ *be such that* $(i, j) \in d$*, then:*

$$\mu_i(N,v,d) = \varphi_i(N, v^d) = \sum_{S\subset N\setminus\{i\}} \frac{s!(n-1-s)!}{n!}\left[v^d(S\cup\{i\}) - v^d(S)\right] =$$

$$= \sum_{S\subset N\setminus\{i,j\}} \frac{(s+1)!(n-2-s)!}{n!}\left[v^d(S\cup\{i,j\}) - v^d(S\cup\{j\})\right] +$$

$$+ \sum_{S\subset N\setminus\{i,j\}} \frac{s!(n-1-s)!}{n!}\left[v^d(S\cup\{i\}) - v^d(S)\right].$$

A symmetrical expression can be obtained for $\mu_j(N, v, d)$*. Then*

$$\mu_i(N,v,d) - \mu_j(N,v,d) =$$

$$= \sum_{S\subset N\setminus\{i,j\}} \frac{(s+1)!(n-2-s)!}{n!}\left[v^d(S\cup\{i,j\}) - v^d(S\cup\{j\})\right] +$$

$$+ \sum_{S\subset N\setminus\{i,j\}} \frac{s!(n-1-s)!}{n!}\left[v^d(S\cup\{i\}) - v^d(S)\right] -$$

$$- \sum_{S\subset N\setminus\{i,j\}} \frac{(s+1)!(n-2-s)!}{n!}\left[v^d(S\cup\{i,j\}) - v^d(S\cup\{i\})\right] -$$

$$- \sum_{S \subset N \setminus \{i,j\}} \frac{s!(n-1-s)!}{n!} \left[v^d(S \cup \{j\}) - v^d(S) \right].$$

All terms different from $v^d(S \cup \{i\})$ or $v^d(S \cup \{j\})$ vanish and thus

$$\mu_i(N,v,d) - \mu_j(N,v,d) =$$

$$\sum_{S \subset N \setminus \{i,j\}} \left[\frac{s!(n-1-s)!}{n!} + \frac{(s+1)!(n-2-s)!}{n!} \right] \left[v^d(S \cup \{i\}) - v^d(S \cup \{j\}) \right] =$$

$$= \sum_{S \subset N \setminus \{i,j\}} \frac{s!(n-2-s)!}{(n-1)!} \left[v^d(S \cup \{i\}) - v^d(S \cup \{j\}) \right]. \tag{4}$$

Since $v^d(S) = v^{d \setminus \{(i,j)\}}(S)$ for all $S \subset N \setminus \{i\}$ or $S \subset N \setminus \{j\}$, it is obvious that expression (4) coincides with $\mu_i(N,v,d \setminus \{(i,j)\}) - \mu_j(N,v,d \setminus \{(i,j)\})$ and thus fairness is proved.

All that is left to prove now is that any allocation rule on $\mathcal{DCS}^N$ that satisfies component efficiency and fairness has to coincide with the digraph Myerson value. We will prove this by induction on the number of arcs in the digraph (N,d). [3] *Let Ψ be an allocation rule on $\mathcal{DCS}^N$ that is component efficient and fair.*

Firstly, consider the digraph communication situation $(N,v,\emptyset) \in \mathcal{DCS}^N$ with no arcs. Then, all nodes are isolated nodes and component efficiency of both Ψ and μ implies that for all $i \in N$, $\Psi_i(N,v,\emptyset) = v^d(i) = v(i) = \mu_i(N,v,\emptyset)$.

Now, let $(N,v,d) \in \mathcal{DCS}^N$ be a communication situation with $|d| = k \geq 1$ arcs, and by the induction hypothesis suppose that Ψ and μ coincide for directed communication situations with less than k arcs. Using this hypothesis and fairness of both Ψ and μ we have that for any arc $(i,j) \in d$:

$$\Psi_i(N,v,d) - \Psi_j(N,v,d) = \Psi_i(N,v,d \setminus \{(i,j)\}) - \Psi_j(N,v,d \setminus \{(i,j)\}) =$$

[3] Recall that Myerson (1977) proved uniqueness of his value for undirected communication situations by induction on the number of edges.

$$= \mu_i(N, v, d \setminus \{(i,j)\}) - \mu_j(N, v, d \setminus \{(i,j)\}) = \mu_i(N, v, d) - \mu_j(N, v, d) \quad (5)$$

It is easily obtained that (5) holds for all i and j that are in the same connected component. This is true because, even if two nodes in a connected component are not connected, they always have a common node in that component to which they are connected. And thus, for $C \in N/d(\gamma)$ there exists a number d_C such that $\Psi_i(N, v, d) - \mu_i(N, v, d) = d_C$ for all $i \in C$. Using component efficiency of both Ψ and μ we have that for a component $C \in N/d$

$0 = \sum_{i \in C} \Psi_i(N, v, d) - \sum_{i \in C} \mu_i(N, v, d) = \sum_{i \in C} d_C = |C| d_C$ *and thus* $d_C = 0$. *Hence:*

$$\Psi(N, v, d) = \mu(N, v, d). \quad \square$$

The next result proves that, in the previous theorem fairness can be replaced by the requirement of balanced contributions. The balanced contributions property for digraph communication situations states that the harm that the isolation of player i can inflict upon player j is the same as the harm that isolation of player j can inflict upon player i.

In order to formalize this definition, given a digraph (N, d), let us denote (N, d_{-k}) for the digraph given by $d_{-k} = \{(i, j) \in d \mid k \notin \{i, j\}\}$.

Definition 4.3 *An allocation rule $\Psi : \mathcal{DCS}^N \to I\!R^n$ satisfies the balanced contributions property if, for all $(N, v, d) \in \mathcal{DCS}^N$ and all $i, j \in N$:*

$$\Psi_j(N, v, d) - \Psi_j(N, v, d_{-i}) = \Psi_i(N, v, d) - \Psi_i(N, v, d_{-j}).$$

Theorem 4.2 *The digraph Myerson value is the unique allocation rule on $\mathcal{DCS}^N$ that satisfies component efficiency and balanced contributions.*

Proof 4.2 *It is already shown in the proof of Theorem 4.1 that the digraph Myerson value satisfies component efficiency.*

To prove that it satisfies balanced contributions, let us observe that if (N, v, d) *is in* $\mathcal{DCS}^N$ *and* $i, j \in N$*, then, for* $S \subset N \setminus \{i, j\}$*, the following holds:* $v^d(S \cup \{i\}) = v^{d_{-j}}(S \cup \{i\})$*,* $v^d(S \cup \{j\}) = v^{d_{-i}}(S \cup \{j\})$ *and* $v^{d_{-j}}(S) = v^{d_{-i}}(S)$*. And thus from (4), we obtain*

$$\mu_i(N, v, d) - \mu_j(N, v, d) =$$

$$= \sum_{S \subset N \setminus \{i,j\}} \frac{s!(n-2-s)!}{(n-1)!} \left[v^{d_{-j}}(S \cup \{i\}) - v^{d_{-i}}(S \cup \{j\}) \right] =$$

$$= \sum_{S \subset N \setminus \{i,j\}} \frac{s!(n-2-s)!}{(n-1)!} \left[v^{d_{-j}}(S \cup \{i\}) - v^{d_{-j}}(S) + v^{d_{-i}}(S) - v^{d_{-i}}(S \cup \{j\}) \right] =$$

$$= \sum_{S \subset N \setminus \{i,j\}} \frac{s!(n-2-s)!}{(n-1)!} \left[v^{d_{-j}}(S \cup \{i\}) - v^{d_{-j}}(S) \right] -$$

$$- \sum_{S \subset N \setminus \{i,j\}} \frac{s!(n-2-s)!}{(n-1)!} \left[v^{d_{-i}}(S \cup \{j\}) - v^{d_{-i}}(S) \right] =$$

$$= \mu_i(N, v, d_{-j}) - \mu_j(N, v, d_{-i}).$$

The last equality can be obtained in the same way we proved (4).

Finally, the proof of uniqueness mimics (using again the induction on the number of arcs in (N, d)*) the corresponding part of the proof of Theorem 4.1 (but applying balanced contributions instead of fairness) and is therefore omitted.* □

5 Final Remarks

1. The purpose of this paper is to give a generalization of the Myerson value and provide axiomatizations through the properties of component efficiency, fairness and balanced contributions. Nevertheless, there are other properties of the classical Myerson value such as, for instance, stability that deserve, we consider, to be analyzed in this new setting. Stability for undirected communication situations means that deleting a link between two players does not increase the payoff of these two players. We can generalize this notion to digraph communication situations as follows.

Definition 5.1 *An allocation rule* $\Psi : \mathcal{DCS}^N \longrightarrow I\!R^n$ *is stable if, for all* $(N, v, d) \in \mathcal{DCS}^N$*, and for any arc* $(i, j) \in d$ *it holds that* $\Psi_k(N, v, d) \geq \Psi_k(N, v, d \setminus \{(i, j)\}), \quad k = i, j.$

In the case of the classical Myerson value defined on $\mathcal{CS}^N$ we know (Myerson, 1977) that a sufficient condition for stability is superadditivity of the initial game. The next example proves that in this new setting superadditivity is not sufficient to guarantee stability.

Example 5.1 *Let us consider the digraph communication situation* $(N, \hat{v}, d)$ *where* $N = \{1, 2, \ldots, 12\}$,

$$v = \sum_{S \subset N,\ s \geq 2} (-1)^s\ 2^{s-2} u_S, \qquad \hat{v} = \sum_{S \subset N,\ s \geq 2,\ T \in \pi(S)} (-1)^s\ 2^{s-2} w_T$$

and (N, d) *is given by* $d = \{(1, 2), (1, 8), (1, 9), (1, 10), (1, 11), (1, 12), (2, 1), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7)\}$*, as represented in Fig. 3.*

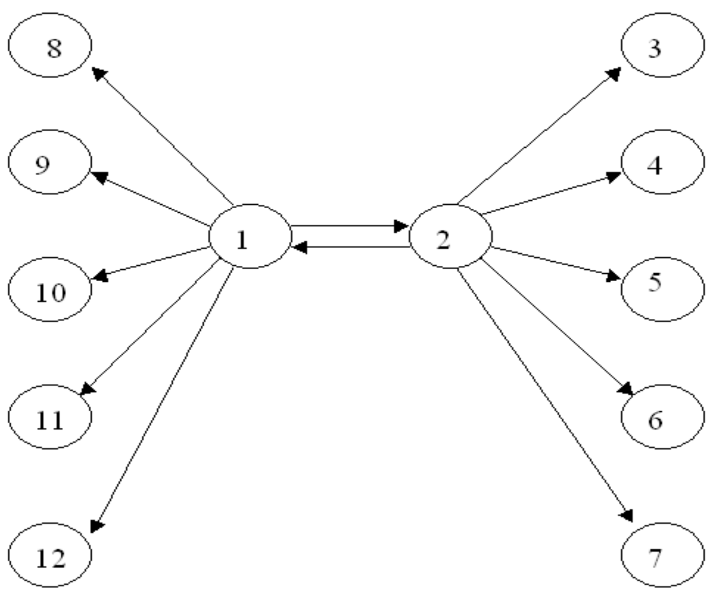

Fig. 3. Digraph in Examples 5.1 and 5.2

It can be argued that $(N, \hat{v})$ is superadditive. Although it is not clear how to define the concept of superadditivity for generalized games, any definition should extend the one for standard games. The generalized game $\hat{v}$ in this example is the transformed game of v, a classical superadditive game and thus, must be considered as superadditive in the generalized setting.

We have that, for $i \neq j$, $i, j \in N$:

$$w^d_{(i,j)} = \begin{cases} \frac{1}{2!} u_{\{i,j\}}, & if \ (i,j) \in d \\ \frac{1}{2!} u_{\{1,2,j\}}, & if \ [i = 1, \ j \in \{3, \ldots, 7\}] \ \ or \ \ [i = 2, \ j \in \{8, \ldots, 12\}] \\ 0, & otherwise \end{cases}$$

$$w^d_{(i,j,k)} = \begin{cases} \frac{1}{3!} u_{\{i,j,k\}}, & if \ \ [i = 1, j = 2] \ \ \ or \ \ [i = 2, j = 1] \\ 0, & otherwise \end{cases}$$

and $w^d_T = 0$ if $T \in \pi(S)$ with $S \subset N$, $4 \leq s \leq 12$. Then, using the previously mentioned additivity property:

$$\mu_1(N, \hat{v}, d) = \left(7 \cdot \frac{1}{2} \ + \ 10 \cdot \frac{1}{3}\right) \ \frac{1}{2!} \ + \ (-2) \cdot 20 \cdot \frac{1}{3} \cdot \frac{1}{3!} = \frac{43}{36}, \ and$$

$$\mu_2(N, \hat{v}, d) = \mu_1(N, \hat{v}, d), \ by \ symmetry.$$

Let us consider now the directed communication situation $(N, \hat{v}, d \setminus \{(1,2)\})$.

As is derived from previous calculations, the digraph Myerson value for players 1 and 2 depend only on the number of ordered coalitions of size 2 and 3 to which they belong and the dividends of these coalitions. Then:

$$\mu_1(N, \hat{v}, d \setminus \{(1,2)\})) = \left(6 \cdot \frac{1}{2} \ + \ 5 \cdot \frac{1}{3}\right) \ \frac{1}{2!} \ + \ (-2) \cdot 5 \cdot \frac{1}{3} \cdot \frac{1}{3!} = \frac{64}{36}.$$

And so, if arc $(1,2)$ is removed, the value for player 1 increases, even though $\hat{v}$ is a superadditive game. Thus, superadditivity does not imply stability in this *setting.*

2. Another open problem is the inheritance of properties in directed communication situations. It is not obvious in which manner properties such as superadditivity, convexity or even solution concepts as the core, should be extended to this framework of generalized games and then, it is difficult to analyze the inheritance of properties.

Nevertheless, it is easy to find some pathologies. Let us look at the superadditivity. Even if we don't know a definition of this property for generalized games, it is obvious that whatever definition is used, this definition must extend the well-known one for TU-games. Then, using Example 5.1 it is easy to prove that superadditivity is not, in general, inherited by the digraph restricted game.

Example 5.2 *Effectively, suppose* $(N, \hat{v}, d)$ *is defined as in Example 5.1.* (N, v) *is a superadditive game (as a TU-game) and consider* A*,* $B \subset N$*,* $A = \{1, 8, 9, 10, 11, 12\}$*;* $B = \{2, 3, 4, 5, 6, 7\}$*. Then*

$$\hat{v}^d(A) = \frac{\Delta^*_{\hat{v}}(1,8) + \Delta^*_{\hat{v}}(1,9) + \Delta^*_{\hat{v}}(1,10) + \Delta^*_{\hat{v}}(1,11) + \Delta^*_{\hat{v}}(1,12)}{2} = \frac{5}{2}.$$

$$\hat{v}^d(B) = \frac{\Delta^*_{\hat{v}}(2,3) + \Delta^*_{\hat{v}}(2,4) + \Delta^*_{\hat{v}}(2,5) + \Delta^*_{\hat{v}}(2,6) + \Delta^*_{\hat{v}}(2,7)}{2} = \frac{5}{2}.$$

$$\hat{v}^d(A \cup B) = \hat{v}^d(A) + \hat{v}^d(B) + \sum_{j=3}^{7} \frac{\Delta^*_{\hat{v}}(1,j)}{2!} + \sum_{j=8}^{12} \frac{\Delta^*_{\hat{v}}(2,j)}{2!} +$$

$$+ \frac{\Delta^*_{\hat{v}}(1,2) + \Delta^*_{\hat{v}}(2,1)}{2!} + \sum_{k=3}^{12} \frac{\Delta^*_{\hat{v}}(1,2,k) + \Delta^*_{\hat{v}}(2,1,k)}{3!} =$$

$$= \hat{v}^d(A) + \hat{v}^d(B) + 5 \cdot \frac{1}{2!} + 5 \cdot \frac{1}{2!} + 2 \cdot \frac{1}{2!} + (-2) \cdot 20 \cdot \frac{1}{3!} =$$

$$= \hat{v}^d(A) + \hat{v}^d(B) - \frac{2}{3} < \hat{v}^d(A) + \hat{v}^d(B).$$

3. There exist other characterizations of the Myerson value that use additional properties such as the superfluous player and player anonymity properties. It could be interesting to analyze the meaning and usefulness of these properties in this new context and to explore if, eventually, that characterizations still holds.

4. Previous results could be extended to a probabilistic setting as it was done, for the case of (undirected) communication situations in Gómez et al. (2007)

5. If we assume that our model introduces an asymmetry between both incident players in a fixed arc, fairness can be viewed as a not very appealing property. Nevertheless, for such models in which the arcs represent communications between players (and then both players are equally important to establish this communication), this fairness property seems desirable.

6. Other future questions could be the exploration of calculation methods for the digraph Myerson value, as in Gómez et al. (2004a and 2004b), and in particular of the restricted game dividends.

7. Moreover, it seems very appealing to study and generalize alternative values for communication situations such as the value proposed by Hamiache (1999) or *the Position Value* (see Borm, Owen and Tijs, 1992).

8. The introduced value also can be used to obtain a centrality measure for actors or nodes in directed social networks, generalizing the one defined in Gómez et al. (2003), for undirected social networks.

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